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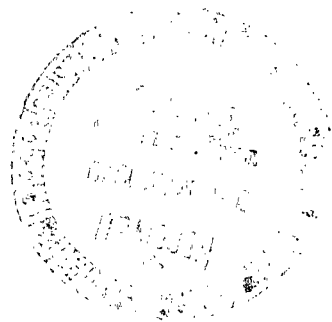


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A FINITE ELEMENT APPROACH
TO THE STRUCTURAL INSTABILITY
OF BEAM COLUMNS, FRAMES, AND ARCHES

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1. REPORT NO. NASA TN D-5782		2. GOVERNMENT ACCESSION NO.		3. RECIPIENT 0132513	
4. TITLE AND SUBTITLE A Finite Element Approach to the Structural Instability of Beam Columns, Frames, and Arches				5. REPORT DATE May 1970	
				6. PERFORMING ORGANIZATION CODE	
7. AUTHOR(S) Jerrell M. Thomas				8. PERFORMING ORGANIZATION REPORT # M-351	
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812				10. WORK UNIT NO. 981-10-10-0000-50-00-08	
				11. CONTRACT OR GRANT NO.	
				13. TYPE OF REPORT & PERIOD COVERED Technical Note	
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546				14. SPONSORING AGENCY CODE	
15. SUPPLEMENTARY NOTES Prepared by Astronautics Laboratory Science and Engineering Directorate					
16. ABSTRACT A nonlinear stiffness matrix for a beam column element subjected to nodal forces and to a uniformly distributed load is developed from the principle of virtual displacements and the bifurcation theory of elastic stability. Three cases of applied load behavior are considered. The buckling load of a uniform circular arch is calculated as an example.					
17. KEY WORDS				18. DISTRIBUTION STATEMENT Unclassified - Unlimited	
19. SECURITY CLASSIF. (of this report) Unclassified		20. SECURITY CLASSIF. (of this page) Unclassified		21. NO. OF PAGES 58	
				22. PRICE* \$3.00	

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151



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ACKNOWLEDGMENT

This analysis was done at the suggestion of Dr. C. V. Smith of Georgia Institute of Technology. Mr. John Key assisted in checking certain portions of the analysis. Mr. Dan Roberts of Computer Sciences Corporation did all of the programming for the example problem. The contributions of these people are gratefully acknowledged.

LIST OF SYMBOLS

Symbol	Meaning
A	Cross sectional area of element
E	Modulus of elasticity of material
F	Force applied to element node
\bar{F}	Force applied to element node referenced to undeformed element coordinate system
\vec{F}	Proportionality constants between forces, F , and lateral load, p
$\tilde{G} = [G]$	Geometric stiffness matrix defined by equation (46)
I	Moment of inertia of element cross section
$\tilde{K} = [K]$	Conventional stiffness matrix defined by equation (45)
$\tilde{K}' = [K']^*$	Total element stiffness matrix (see equation (44))
$\bar{K} = [\bar{K}]$	Element stiffness matrix in global coordinate system
$\tilde{K}_0 = [K_0]$	Nonlinear stiffness matrix defined by equation (91)
L	Length of element
$\tilde{L} = [L]$	Load behavior stiffness matrix
$\tilde{L}_F = [L_F]$	Nodal force behavior stiffness matrix
$\tilde{L}_p = [L_p]$	Lateral load behavior stiffness matrix
l	Distance from element node to point through which the F_x nodal forces are directed

LIST OF SYMBOLS (Continued)

Symbol	Meaning
M	Bending moment applied to element node
$\bar{M} = M$	Bending moment applied to element node referenced to undeformed element coordinate system
m, n	Unit vectors in element coordinate system
\bar{m}, \bar{n}	Unit vectors in global coordinate system
p	Lateral load applied to element
Q	Nodal force or moment
$\underline{Q} = \{Q\}$	Column vector of nodal forces and moments
$\bar{\underline{Q}} = \{\bar{Q}\}$	Column vector of nodal forces and moments referenced to global coordinate system
q	Nodal displacement or rotation
$\underline{q} = \{q\}$	Column vector of nodal displacements
$\bar{\underline{q}} = \{\bar{q}\}$	Column vector of nodal displacements in global coordinate system
R	Distance from element centerline to point through which lateral forces are directed; also radius of circular arch
$\underline{T} = [T]$	Coordinate transformation matrix defined by equation (81)
U	Strain energy of the element
u	Displacement in x direction
\bar{u}	Displacement in \bar{x} direction

LIST OF SYMBOLS (Continued)

Symbol	Meaning
v	Volume of the element
w	Displacement in z direction
\bar{w}	Displacement in \bar{z} direction
x, z	Element coordinates
\bar{x}, \bar{z}	Global coordinates
α	Constant coefficient
β	Angle between element and global coordinate systems
γ	Displacement functions defined by equation (6)
$\delta U, \delta V$	Change in strain energy, virtual work of external forces
ϵ_x	Strain of element in x direction at any point
ϵ_{xx}	Strain of element midsurface in x direction
θ	Rotation of element node
κ	Curvature of beam element centerline
σ_x	Stress in x direction at any point in the element
ϕ	Displacement functions defined by equation (5)
ω	Unit rotation vector

LIST OF SYMBOLS (Concluded)

Symbol	Meaning
$\left. \begin{matrix} [\gamma_k \gamma_j \gamma_i] \\ [\phi_i \phi_j \phi_k] \\ [\phi_i \phi_k \gamma_j] \\ [\phi_k \phi_j \gamma_i] \end{matrix} \right\}$	Matrices defined by equations (55) through (61)
Subscript	Meaning
1, 2	Denotes left and right nodes respectively of the elements; also denotes a particular displacement function, γ
1, 2, 3, 4	Denotes a particular ϕ displacement function
a, b, c, d	Denotes a particular element of a general structure
i, j, k	Denotes a particular node of the element or one of the displacement functions, γ or ϕ
<u>i</u>	Indicates no summation on double subscript, i
x, z	Direction of the coordinates x and z
$\cdot, \ddot{x}; \cdot, \ddot{xx}$	First and second derivatives with respect to x
Superscript	Meaning
0	Denotes displacements of the structure just prior to buckling
1	Denotes displacements of the structure just after buckling
-1	Inverse of a matrix
T	Transpose of a matrix

A FINITE ELEMENT APPROACH TO THE STRUCTURAL INSTABILITY OF BEAM COLUMNS, FRAMES, AND ARCHES

SUMMARY

From the principle of virtual displacements and the bifurcation theory of elastic stability a stiffness matrix is developed for a beam column element with shear, moment, and axial load applied to the ends (nodes) of the element and a uniformly distributed load applied along the span of the element. The stiffness matrix developed relates the forces applied at the nodes to the displacements of the nodes and is a function of the magnitude of the applied load. Three types of behavior of the applied loads are considered as separate cases. These are: the loads remain normal to the deformed element; the loads remain parallel to their original direction; and, the loads remain directed toward a fixed point.

A method is given for using the element stiffness matrix to predict the buckling load for a structure which may be represented by beam column elements. As an example, the buckling load of an arch for each of the three load-behavior cases is calculated and compared to known solutions.

INTRODUCTION

In recent years interest in the solution of structural problems by the finite element method has greatly increased. This interest can be attributed to the fact that the digital computer has made possible the solution of a great number of complex, yet practical, problems by this method.

Those active in publishing results in the field include a group at the University of Washington and The Boeing Company [1-9], J. H. Argyris at the University of London [10, 11], and R. H. Gallagher and his associates at Bell Aerosystems Company [12-14].

Basically the method consists of representing an actual structure by an idealized structure made up of elements for which relationships between the forces applied to points (nodes) on the boundary of the element and the displacements of the nodes is known. This relationship is conveniently represented by

an element stiffness matrix. If all of the element stiffness matrices are transformed to a common (global) coordinate system, the sum of forces applied to the elements meeting at each node is equated to the externally applied force at that node, and the displacements of elements which meet at each node are equated, a set of equations relating the externally applied forces to the nodal displacements results. When this set is expressed in matrix form, the matrix relating the externally applied forces to the nodal displacements is called the master stiffness matrix. Then for any given combination of known applied external forces and nodal displacements the above relationships may be algebraically manipulated to determine internal loads and displacements throughout the structure.

Once a sufficient variety and quality of element stiffness matrices are available to idealize a particular structure, the remainder of the above procedure to solve a particular problem becomes tedious but quite mechanical. Hence the advantage of the computer in solving such problems is obvious.

Although the use of such techniques in solving complex structural problems is now an everyday occurrence, a great deal of developmental activity in the field still persists for several reasons. First, there is no such thing as a correct stiffness matrix for a particular element which excludes other stiffness matrices for that same element as incorrect. A number of correct stiffness matrices may be developed for the same element if different assumptions are made for the derivation. Each of these matrices may have its own advantages for some particular application and all of them may converge to identical answers if the structure to be analyzed is broken into small enough elements. It is the search for the most advantageous element for particular applications or for all-inclusive elements which gives rise to much of the current research.

Second, many derivations have been based on geometrical considerations only and have sometimes led to incorrect stiffness matrices or matrices with poor convergence characteristics. The recent trend has been to base derivations on basic elasticity principles and to use more care in assuring that adjacent element deformations conform to each other not only at the nodes but along the complete boundary [5, 15-19].

Third, extension of the finite element techniques to include a larger class of problems has received a great deal of recent attention. In particular, the solution of nonlinear problems including large deformations and structural instability has been of interest [1-3, 5, 8, 9, 13-15, 17, 20].

This report was motivated to some extent by all three of the above considerations. The objective was to develop a beam column element, beginning with fundamental principles, which would account for the applied load behavior and could be used for structural stability analysis.

It is assumed that the reader is familiar with the conventional matrix and index notation used throughout the report. Since there is no stringent space limitation, the work is presented in an unusual amount of detail with the presumption that the development could serve as a guide for the development of curved or three-dimensional elements.

GENERAL THEORY

Basic Assumptions and Limitations

The usual Euler-Bernoulli beam assumptions are made for developing the beam column element stiffness matrix. These are basic engineering assumptions and may be listed as follows:

1. The material of the element is homogeneous and isotropic.
2. Plane sections remain plane after bending.
3. The stress-strain curve is identical in tension and compression.
4. Hooke's law holds.
5. The effect of transverse shear is negligible.
6. The deflections are small compared to the cross sectional dimensions.
7. The loads act in a single plane passing through a principal axis of inertia of the cross section.
8. No initial curvature of the element exists.
9. No local type of instability will occur within the element.
10. Loads are applied quasi-statically.

The above assumptions limit the application of results to structures that lie in a plane and are loaded in the same plane. Out-of-plane and torsional buckling are not treated.

Method of Analysis

The work done during a virtual displacement of the element is equated to zero, as presented by Hoff in Reference 21 and derived more rigorously by Langhaar in Reference 22, to obtain the equilibrium equations for the element. The bifurcation concept of elastic stability as presented by Novozhilov in Reference 23, Chapter V, is used to postulate that two possible sets of displacements which satisfy the equilibrium equations may exist under the same magnitude of external load if the magnitude is such that a structure is unstable. Each of these sets of displacements is substituted in turn into the equilibrium equations. The resulting sets of equations are combined to obtain a relationship between the nodal forces and the nodal displacements during buckling. When placed in matrix form this relationship becomes

$$\{Q^1\} = [K] + [G] + [L] \{q^1\} = \{0\} \quad (1)$$

where \tilde{Q}^1 is the column matrix of nodal forces, \tilde{q}^1 is the column matrix of nodal displacements, \tilde{K} is the usual beam element stiffness matrix, \tilde{G} is the geometric stiffness matrix, and \tilde{L} is the load behavior stiffness matrix. \tilde{G} and \tilde{L} both contain the magnitude of external load as a factor. Thus an element stiffness matrix can be derived which is a function of the applied load including the applied load behavior.

A number of such elements may be combined to represent a particular structure, so that equation (1) applies to the entire structure and $\tilde{K} + \tilde{G} + \tilde{L}$ is the master stiffness matrix. Boundary conditions may be applied to reduce the size of the master stiffness matrix. For instability to exist the determinant of the master stiffness matrix must vanish. Hence, an eigenvalue problem is formulated where the eigenvalues are the magnitudes of the applied load at which the structure is unstable.

Development of Element Stiffness Matrix

Description of Element. Consider a beam column element as shown in Figure 1 which is subjected to nodal forces and moments and a uniformly distributed load, p , applied along its length. It is desired to determine a stiffness matrix for the element which may be used to calculate the stability of structures made up of such elements. The stiffness matrix is to account for the fact that the components of the nodal forces or distributed load, or both, may be a function of the element displacements. Three cases are to be considered: the loads remain normal to the deformed element; the loads remain parallel to their original direction; and, the loads remain directed toward a fixed point.

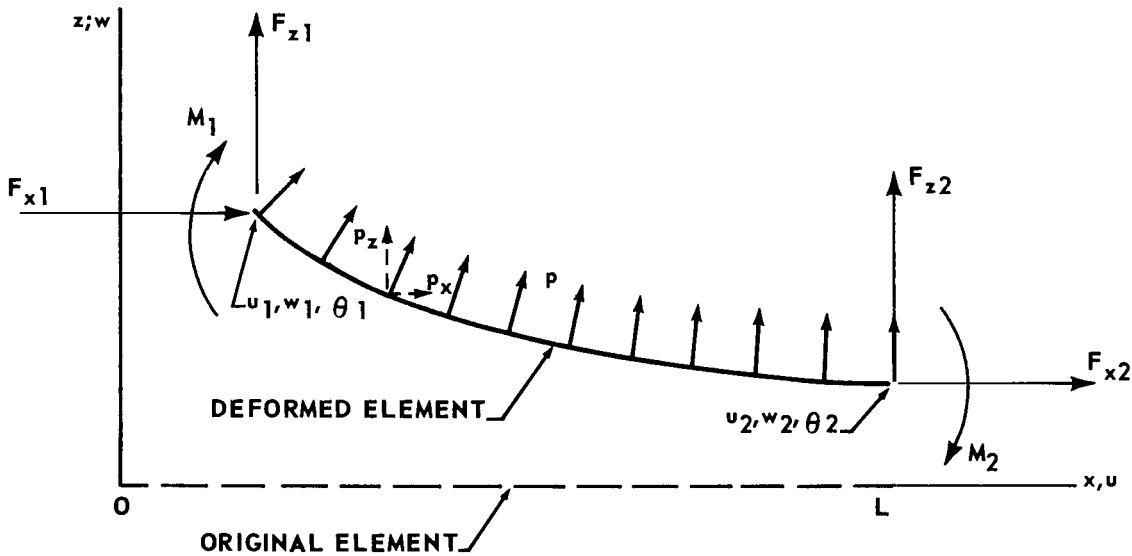


FIGURE 1. BEAM COLUMN ELEMENT

Displacement Functions. If it is assumed that the lateral displacement of the beam element of Figure 1 may be represented by

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad (2)$$

and the longitudinal displacement is given by

$$u = \alpha_5 + \alpha_6 x \quad , \quad (3)$$

then the six unknown constants, α_i , may be determined in terms of the nodal displacements from the element boundary conditions,

$$\begin{aligned}
 w &= w_1 \quad \text{at } x = 0 \\
 w,_{x} &= -\theta_1 \quad \text{at } x = 0 \\
 w &= w_2 \quad \text{at } x = L \\
 w,_{x} &= -\theta_2 \quad \text{at } x = L \\
 u &= u_1 \quad \text{at } x = 0 \\
 u &= u_2 \quad \text{at } x = L \quad .
 \end{aligned} \tag{4}$$

The resulting displacement functions are

$$\begin{aligned}
 w &= w_1 \left(1 - 3 \frac{x^2}{L^2} + 2 \frac{x^3}{L^3} \right) + w_2 \left(3 \frac{x^2}{L^2} - 2 \frac{x^3}{L^3} \right) \\
 &\quad + \theta_1 \left(-x + 2 \frac{x^2}{L} - \frac{x^3}{L^2} \right) + \theta_2 \left(\frac{x^2}{L} - \frac{x^3}{L^2} \right) \\
 &= w_1 \phi_1(x) + w_2 \phi_2(x) + \theta_1 \phi_3(x) + \theta_2 \phi_4(x) \\
 &= w_i \phi_i(x)
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 u &= u_1 \left(1 - \frac{x}{L} \right) + u_2 \left(\frac{x}{L} \right) \\
 &= u_1 \gamma_1(x) + u_2 \gamma_2(x) \\
 &= u_i \gamma_i(x) \quad .
 \end{aligned} \tag{6}$$

These displacement functions have been used by others for the beam column element (see Reference 5, for example). However, it should be mentioned that an inconsistency arises when the cubic function is used for the lateral displacement of the present element which has a distributed load. This is illustrated by taking the third derivative of w , which should be the equation for the shear load on the element, and observing that a constant results. But the element has a distributed load, and the shear should obviously be a linear function, not a constant. Since the change in shear over the length of an element becomes negligible in the limit as the element becomes smaller and smaller, this inconsistency is not unacceptable and it will be seen that adequate results are obtained. The correct fourth order function could not be used for w because there is no other boundary condition available for evaluating another constant in equation (2). This will be discussed further in the conclusions.

Development of Equilibrium Equations. The element equilibrium equations will now be developed from the principle of virtual displacements,

$$\delta U - \delta V = 0 \quad (7)$$

where δU is the change in strain energy and δV is the work done by the external forces during a virtual displacement.

For the uniaxial state of stress assumed to exist in a beam the change in strain energy is given by

$$\delta U = \int_V \sigma_x \delta \epsilon_x dv \quad (8)$$

Here, ϵ_x is the strain of the beam at any point of the cross section and is given by

$$\epsilon_x = \epsilon_{xx} + z\kappa_{xx} \quad (9)$$

where ϵ_{xx} is the strain of the beam mid-surface, z is the distance of the point in the beam from the mid-surface, and

$$\kappa_{xx} = w_{,xx} \quad (10)$$

is the beam curvature. The nonlinear strain-displacement relationship

$$\epsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} u_{,x}^2 \quad (11)$$

is used for the mid-surface strain. The $u_{,x}^2$ term in this expression is not known to have been used in previous derivations of nonlinear beam element stiffness matrices. Substituting equations (9), (10), and (11) into equation (8),

$$\delta U = \int_A \int_L E \epsilon_{xx} \delta \epsilon_{xx} dx dA = \int_A \int_L E (\epsilon_{xx} + z \kappa_{xx}) (\delta \epsilon_{xx} + z \delta \kappa_{xx}) dx dA \quad (12)$$

$$\delta U = \int_A \int_L \left[E \epsilon_{xx} \delta \epsilon_{xx} + E z \epsilon_{xx} \delta \kappa_{xx} + E z \kappa_{xx} \delta \epsilon_{xx} + E z^2 \kappa_{xx} \delta \kappa_{xx} \right] dx dA$$

$$\int_A z dA = 0, \quad \int_A z^2 dA = I$$

$$\delta U = \int_L \left[A E \epsilon_{xx} \delta \epsilon_{xx} + E I \kappa_{xx} \delta \kappa_{xx} \right] dx \quad (13)$$

where

$$\delta \epsilon_{xx} = \delta u_{,x} + \delta \left(\frac{1}{2} w_{,x}^2 \right) + \delta \left(\frac{1}{2} u_{,x}^2 \right) = \delta u_{,x} + w_{,x} \delta w_{,x} + u_{,x} \delta u_{,x}$$

$$\delta \kappa_{xx} = \delta w_{,xx}$$

$$\delta U = \int_L \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} u_{,x}^2 \right) \left(\delta u_{,x} + w_{,x} \delta w_{,x} + u_{,x} \delta u_{,x} \right) + EI w_{,xx} \delta w_{,xx} \right] dx$$

$$\delta U = \int_L \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2 \right) \delta u_{,x} + EA \left(u_{,x} w_{,x} \right) \delta w_{,x} + EI w_{,xx} \delta w_{,xx} \right] dx$$

+ higher order terms. (14)

The virtual work of the external forces acting on the element is given by:

$$\delta V = \bar{F}_{xi} \delta u_i + \bar{F}_{zi} \delta w_i + \bar{M}_i \delta \theta_i + \int_L \left[p_z \delta w + p_x \delta u \right] dx \quad (15)$$

where the force components depend upon the load behavior. Three load behavior cases will be considered.

Case I. Loads Remain Normal to the Deformed Element (Fig. 2)

$$\begin{aligned} p_z &= p \\ p_x &= -p w_{,x} \\ \bar{F}_{xi} &= F_{xi} + F_{zi} \theta_i \\ \bar{F}_{zi} &= F_{zi} - F_{xi} \theta_i \\ \bar{M}_i &= M_i \end{aligned} \quad (16)$$

Then

$$\begin{aligned} \delta V &= \left(F_{xi} + F_{zi} \theta_i \right) \delta u_i + \left(F_{zi} - F_{xi} \theta_i \right) \delta w_i + M_i \delta \theta_i \\ &+ \int_L \left[p \delta w - p w_{,x} \delta u_{,x} \right] dx \end{aligned} \quad (17)$$

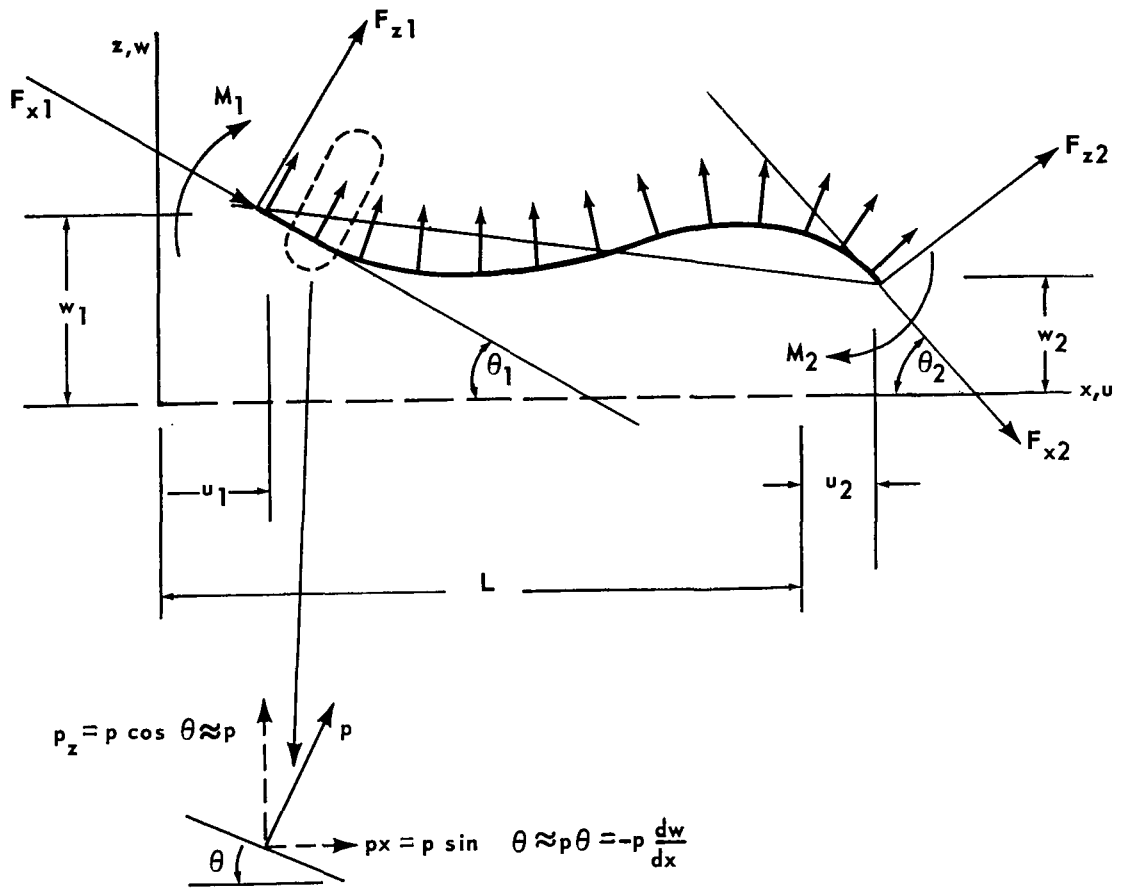


FIGURE 2. LOADS REMAIN NORMAL TO ELEMENT

Case II. Loads Remain Parallel to Their Original Direction (Fig. 3)

$$p_z = p$$

$$p_x = 0$$

$$\bar{F}_{xi} = F_{xi}$$

$$\bar{F}_{zi} = F_{zi}$$

$$\bar{M}_i = M_i$$

(18)

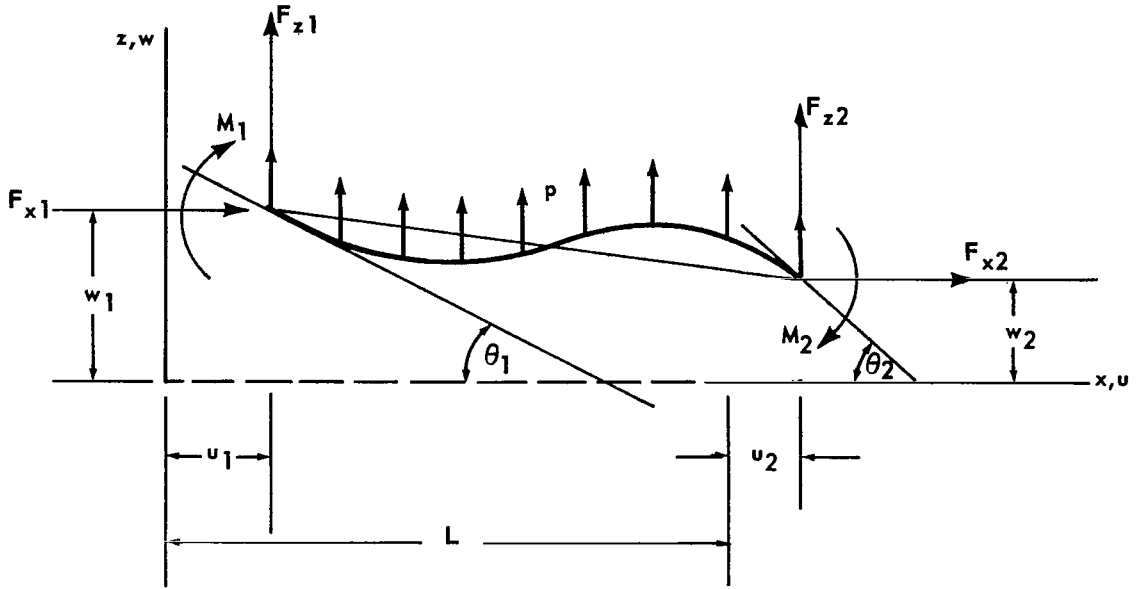


FIGURE 3. LOADS REMAIN PARALLEL TO ORIGINAL DIRECTION

$$\delta V = F_{xi} \delta u_i + F_{zi} \delta w_i + M_i \delta \theta_i + \int_L p \delta w dx \quad (19)$$

Case III. Loads Remain Directed Toward a Fixed Point (Fig. 4)

$$p_z = p$$

$$p_x = \frac{pu}{R}$$

$$\bar{F}_{xi} = F_{xi} + F_{zi} \frac{u_i}{R}$$

$$\bar{F}_{zi} = F_{zi} - F_{xi} \frac{w_i}{l}$$

$$\bar{M}_i = M_i \quad (20)$$

$$\begin{aligned}
\delta V = & \left(F_{xi} + F_{zi} \frac{u_i}{R} \right) \delta u_i + \left(F_{zi} - F_{xi} \frac{w_i}{l} \right) \delta w_i \\
& + M_i \delta \theta_i + \int_0^L \left[p \delta w + \frac{pu}{R} \delta u \right] dx \quad .
\end{aligned} \tag{21}$$

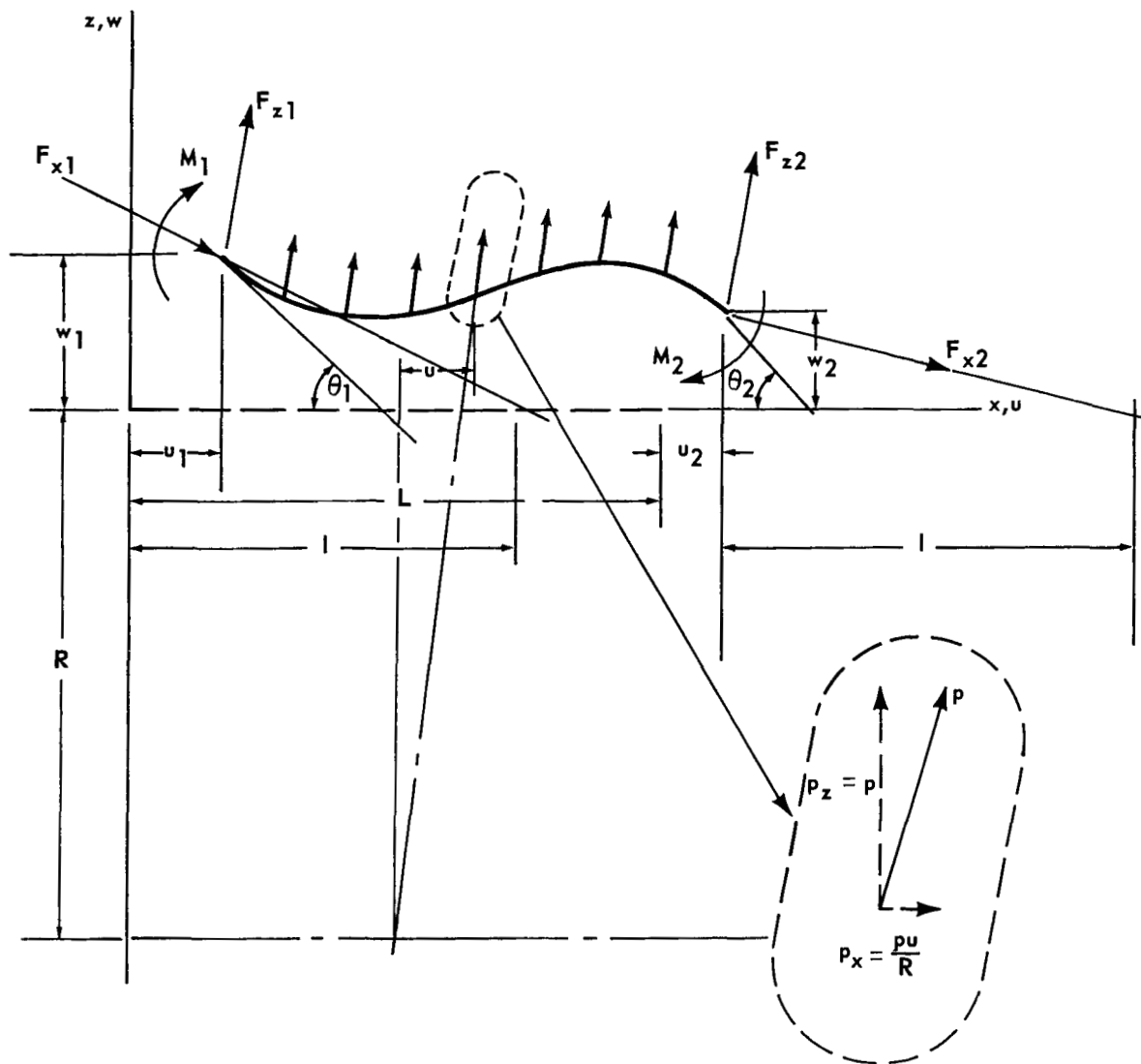


FIGURE 4. LOADS REMAIN DIRECTED TOWARD A FIXED POINT

Substituting equations (14), (17), (19), and (21) into equation (7), we obtain the following results:

Case I.

$$\begin{aligned} & \int_L \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2 \right) \delta u_{,x} + EA \left(u_{,x} w_{,x} \right) \delta w_{,x} \right. \\ & \left. + EI w_{,xx} \delta w_{,xx} \right] dx = \left(F_{xi} + F_{zi} \theta_i \right) \delta u_i + \left(F_{zi} - F_{xi} \theta_i \right) \delta w_i \\ & + M_i \delta \theta_i + \left[\int_L p \delta w - p w_{,x} \delta u \right] dx \end{aligned} \quad (22)$$

Case II.

$$\begin{aligned} & \int_L \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2 \right) \delta u_{,x} + EA \left(u_{,x} w_{,x} \right) \delta w_{,x} \right. \\ & \left. + EI w_{,xx} \delta w_{,xx} \right] dx = F_{xi} \delta u_i + F_{zi} \delta w_i + M_i \delta \theta_i + \int_L p \delta w dx \end{aligned} \quad (23)$$

Case III.

$$\begin{aligned} & \int_L \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2 \right) \delta u_{,x} + EA \left(u_{,x} w_{,x} \right) \delta w_{,x} + EI w_{,xx} \delta w_{,xx} \right] dx \\ & = \left(F_{xi} + F_{zi} \frac{u_i}{R} \right) \delta u_i + \left(F_{zi} - F_{xi} \frac{w_i}{l} \right) \delta w_i + M_i \delta \theta_i \\ & + \int_L \left[p \delta w + \frac{pu}{R} \delta u \right] dx \end{aligned} \quad (24)$$

From the displacement functions, equations (5) and (6)

$$\begin{aligned} u_{,x} &= u_i \gamma_{i,x} \\ w_{,x} &= w_i \phi_{i,x} \\ w_{,xx} &= w_i \phi_{i,xx} \end{aligned} \quad (25)$$

And the above variations can be written,

$$\begin{aligned}
\delta u &= \gamma_i \delta u_i \\
\delta u_{,x} &= \gamma_{i,x} \delta u_i \\
\delta w &= \phi_i \delta w_i \\
\delta w_{,x} &= \phi_{i,x} \delta w_i \\
\delta w_{,xx} &= \phi_{i,xx} \delta w_i .
\end{aligned} \tag{26}$$

Upon substituting equations (25) and (26) into equations (22) through (24) and recalling that $\theta_i = w_{i+2}$, the following equations are obtained:

Case I.

$$\begin{aligned}
&\int_L \left\{ EA \left[\left(u_j \gamma_j + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \right) \gamma_{i,x} \delta u_i \right. \right. \\
&\quad \left. \left. + \left(u_j \gamma_{j,x} w_k \phi_{k,x} \right) \phi_{i,x} \delta w_i \right] + EI w_j \phi_{i,xx} \phi_{j,xx} \delta w_i \right\} dx \\
&\quad - \left(F_{xi} + F_{zi} w_{i+2} \right) \delta u_i - \left(F_{zi} - F_{xi} w_{i+2} \right) \delta w_i \\
&\quad + M_i \delta w_{i+2} - \int_L \left[p \phi_i \delta w_i - p w_j \phi_{j,x} \gamma_i \delta u_i \right] dx = 0
\end{aligned} \tag{27}$$

Case II.

$$\begin{aligned}
&\int_L \left\{ EA \left[\left(u_j \gamma_{j,x} + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \right) \gamma_{i,x} \delta u_i \right. \right. \\
&\quad \left. \left. + \left(u_j \gamma_{j,x} w_k \phi_{k,x} \right) \phi_{i,x} \delta w_i \right] + EI w_j \phi_{i,xx} \phi_{j,xx} \delta w_i \right\} dx \\
&\quad - F_{xi} \delta u_i - F_{zi} \delta w_i - M_i \delta w_{i+2} - \int_L p \phi_i \delta w_i dx = 0
\end{aligned} \tag{28}$$

Case III.

$$\begin{aligned}
& \int_L \left\{ EA \left[\left(u_j \gamma_{j,x} + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \right) \gamma_{i,x} \delta u_i \right. \right. \\
& \left. \left. + \left(u_j \gamma_{j,x} w_k \phi_{k,x} \right) \phi_{i,x} \delta w_i \right] + EI w_j \phi_{i,xx} \phi_{j,xx} \delta w_i \right\} dx \\
& - \left(F_{xi} + F_{zi} \frac{u_i}{R} \right) \delta u_i - \left(F_{zi} - F_{xi} \frac{w_i}{l} \right) \delta w_i - M_i \delta w_{i+2} \\
& - \int_L \left(p \phi_i \delta w_i + \frac{p}{R} u_j \gamma_j \gamma_i \delta u_i \right) dx = 0
\end{aligned} \tag{29}$$

For independent virtual displacements, δu_i and δw_i , the equilibrium equations are obtained from equations (27) through (29).

Case I.

$$\begin{aligned}
& \int_L \left[EA \left(u_j \gamma_{j,x} \gamma_{i,x} + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} \gamma_{i,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) \right. \\
& \left. + p w_j \phi_{j,x} \gamma_i \right] dx - Q_{xi} - F_{zi} w_{i+2} = 0
\end{aligned} \tag{30}$$

$$\begin{aligned}
& \int_L \left[EA u_j \gamma_{j,x} w_k \phi_{k,x} \phi_{i,x} + EI w_j \phi_{i,xx} \phi_{j,xx} - p(\phi_i) \right] dx \\
& - Q_{zi} + F_{xi} w_{i+2} = 0
\end{aligned} \tag{31}$$

Case II.

$$\begin{aligned}
& \int_L \left[EA \left(u_j \gamma_{j,x} \gamma_{i,x} + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} \gamma_{i,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) \right] dx \\
& - Q_{xi} = 0
\end{aligned} \tag{32}$$

$$\int_L \left(EA u_j \gamma_{j,x} w_k \phi_{k,x} \phi_{i,x} + EI w_j \phi_{i,xx} \phi_{j,xx} - p \phi_i \right) dx - Q_{zi} = 0 \quad (33)$$

Case III.

$$\int_L \left[EA \left(u_j \gamma_{j,x} \gamma_{i,x} + \frac{1}{2} w_k w_j \phi_{k,x} \phi_{j,x} \gamma_{i,x} + \frac{3}{2} u_k u_j \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) - \frac{p}{R} u_j \gamma_j \gamma_i \right] dx - Q_{xi} - F_{zi} \frac{u_i}{R} = 0 \quad (34)$$

$$\int_L \left(EA u_j \gamma_{j,x} w_k \phi_{k,x} \phi_{i,x} + EI w_j \phi_{i,xx} \phi_{j,xx} - p \phi_i \right) dx - Q_{zi} + F_{xi} \frac{w_i}{l} = 0 \quad (35)$$

Equations (30) through (35) are the final equilibrium equations for the beam column element.

Bifurcation Theory of Instability. Let a solution of the equilibrium equations be u_i^0, w_i^0 . According to the bifurcation concept of instability, at the instability load magnitude there is another set of displacements, arbitrarily close to the first set, which also satisfies the equations of equilibrium. Denote this second set of nodal displacements by $u_i^0 + u_i^1, w_i^0 + w_i^1$.

Upon substituting this new solution into the equilibrium equations, there results for Case I

$$\begin{aligned} & \int_L \left\{ EA \left[\left(u_j^0 + u_j^1 \right) \gamma_{j,x} \gamma_{i,x} + \frac{1}{2} \left(w_k^0 + w_k^1 \right) \left(w_j^0 + w_j^1 \right) \phi_{k,x} \phi_{j,x} \gamma_{i,x} \right. \right. \\ & \left. \left. + \frac{3}{2} \left(u_k^0 + u_k^1 \right) \left(u_j^0 + u_j^1 \right) \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right] + p \left(w_j^0 + w_j^1 \right) \phi_{j,x} \gamma_i \right\} dx \\ & - Q_{xi} - F_{zi} \left(\frac{w_i^0}{l} + \frac{w_i^1}{l} \right) = 0 \end{aligned} \quad (36)$$

$$\int_L \left[EA \left(u_j^0 + u_j^1 \right) \left(w_k^0 + w_k^1 \right) \gamma_{j,x} \phi_{k,x} \phi_{i,x} + EI \left(w_j^0 + w_j^1 \right) \phi_{i,xx} \phi_{j,xx} - p \phi_i \right] dx - Q_{zi} + F_{xi} \left(w_{\underline{i}+2}^0 + w_{\underline{i}+2}^1 \right) = 0 \quad (37)$$

Similar results are obtained for Cases II and III.

If these equations are expanded, if the q_i^0 state terms (which themselves satisfy the equations) are canceled, and if only linear terms in the arbitrarily small q_i^1 state are retained, the following sets of equations result.

Case I.

$$\int_L \left[EA \left(u_j^1 \gamma_{j,x} \gamma_{i,x} + w_j^1 w_k^0 \phi_{k,x} \phi_{j,x} \gamma_{i,x} + 3u_k^0 u_j^1 \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) + p w_j^1 \phi_{j,x} \gamma_i \right] dx - F_{zi} \theta_{\underline{i}}^1 = 0 \quad (38)$$

$$\int_L \left[EA \left(u_j^1 w_k^0 \phi_{i,x} \phi_{k,x} \gamma_{j,x} + w_j^1 u_k^0 \phi_{i,x} \phi_{j,x} \gamma_{k,x} \right) + EI w_j^1 \phi_{i,xx} \phi_{j,xx} \right] dx + F_{xi} \theta_{\underline{i}}^1 = 0 \quad (39)$$

Case II.

$$\int_L \left[EA \left(u_j^1 \gamma_{j,x} \gamma_{i,x} + w_j^1 w_k^0 \phi_{k,x} \phi_{j,x} \gamma_{i,x} + 3u_k^0 u_j^1 \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) \right] dx = 0 \quad (40)$$

$$\int_L \left[EA \left(u_j^1 w_k^0 \phi_{i,x} \phi_{k,x} \gamma_{j,x} + w_j^1 u_k^0 \phi_{i,x} \phi_{j,x} \gamma_{k,x} \right) + EI w_j^1 \phi_{i,xx} \phi_{j,xx} \right] dx = 0 \quad (41)$$

Case III.

$$\int_L \left[EA \left(u_j^1 \gamma_{j,x} \gamma_{i,x} + w_j^1 w_k^0 \phi_{k,x} \phi_{j,x} \gamma_{i,x} + 3 u_k^0 u_j^1 \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} \right) - \frac{p}{R} u_j^1 \gamma_{j,x} \gamma_{i,x} \right] dx - F_{zi} \frac{u_i}{R} = 0 \quad (42)$$

$$\int_L \left[EA \left(u_j^1 w_k^0 \phi_{i,x} \phi_{k,x} \gamma_{j,x} + w_j^1 u_k^0 \phi_{i,x} \phi_{j,x} \gamma_{k,x} \right) + EI w_j^1 \phi_{i,xx} \phi_{j,xx} \right] dx + F_{xi} \frac{w_i^1}{1} = 0 \quad (43)$$

These equations may be expressed in matrix notation for all three cases as follows:

$$\left[[K] + [G] + [L_p] + [L_F] \right] \{q^1\} = \{0\} \quad (44)$$

or

$$[K^1] \{q^1\} = \{0\}$$

where

$$[K^1] = [K] + [G] + [L_p] + [L_F]$$

$$[K] \{q^1\} = \begin{bmatrix} \int_L EA \gamma_{j,x} \gamma_{i,x} dx & 0 \\ \vdots & \vdots \\ 0 & \int_L EI \phi_{i,xx} \phi_{j,xx} dx \end{bmatrix} \begin{Bmatrix} u_j^1 \\ \vdots \\ w_j^1 \end{Bmatrix} \quad (45)$$

$$[G] \{q^1\} = \left[\begin{array}{c|c} \int_L 3EA u_k^0 \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} dx & \int_L EA w_k^0 \phi_{k,x} \phi_{j,x} \gamma_{i,x} dx \\ \hline \int_L EA w_k^0 \phi_{i,x} \phi_{k,x} \gamma_{j,x} dx & \int_L EA u_k^0 \phi_{i,x} \phi_{j,x} \gamma_{k,x} dx \end{array} \right] \left\{ \begin{array}{c} u_j^1 \\ w_j^1 \end{array} \right\} \quad (46)$$

for all cases.

The L matrices are different for each case.

Case I.

$$[L_p] \{q^1\} = \left[\begin{array}{c|c} 0 & \int_L p \phi_{j,x} \gamma_i dx \\ \hline 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_j^1 \\ w_j^1 \end{array} \right\} \quad (47)$$

$$[L_F] \{q^1\} = \left\{ \begin{array}{c} -F_{zi} \theta_i^1 \\ \hline F_{xi} \theta_i^1 \end{array} \right\} \quad (48)$$

Case II.

$$[L_p] = [L_F] = [0] \quad (49)$$

Case III.

$$[L_p] \{q^1\} = \left[\begin{array}{c|c} -\int_L \frac{p}{R} \gamma_j \gamma_i dx & 0 \\ \hline 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_j^1 \\ w_j^1 \end{array} \right\} \quad (50)$$

$$[L_F]\{q^1\} = \begin{pmatrix} F_{zi} & \frac{u_i^1}{R} \\ \text{---} & \text{---} \\ F_{xi} & \frac{w_i^1}{l} \end{pmatrix} \quad (51)$$

Conventional Stiffness Matrix \tilde{K} . Equation (45) represents the conventional beam column element stiffness matrix, which is suitable for use when there is no interest in nonlinear effects or instability. The elements of this matrix may be derived from the general expression, equation (45). For example,

$$\gamma_1 = \left(1 - \frac{x}{L}\right), \quad \phi_1 = \left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right)$$

from equations (5) and (6). Then,

$$\gamma_{1,x} = -\frac{1}{L}, \quad \phi_{1,x} = -6\frac{x}{L^2} + 6\frac{x^2}{L^3}, \quad \phi_{1,xx} = -\frac{6}{L^2} + \frac{12x}{L^3}.$$

Therefore,

$$K_{11} = \int_0^L EA \gamma_{1,x} \gamma_{1,x} dx = \int_0^L EA \left(\frac{1}{L^2}\right) dx = \frac{AE}{L}$$

$$K_{33} = \int_0^L EI \phi_{1,xx} \phi_{1,xx} dx = \int_0^L EI \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)^2 dx = \frac{12EI}{L^3}$$

Other elements of \tilde{K} are obtained similarly to yield:

$$\begin{aligned}
& \begin{matrix} & u_1 & u_2 & w_1 & w_2 & \theta_1 & \theta_2 \\ [K] = & \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI}{L^3} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & 0 & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ 0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} & \begin{matrix} F_{x1} \\ F_{x2} \\ F_{z1} \\ F_{z2} \\ M_1 \\ M_2 \end{matrix} \end{matrix} \\
& \hspace{15em} (52)
\end{aligned}$$

Geometric Stiffness Matrix \tilde{G} . Equation (46) is the general form of the geometric stiffness matrix which accounts for the effect of loads existing in the element on the stiffness of the element. For example, it is well known that the axial load in a beam column has an appreciable effect on the lateral stiffness. The geometric stiffness matrix is an adjustment to the conventional stiffness matrix to account for such effects. Such matrices have also been referred to in the literature as stability coefficient matrices and incremental stiffness matrices. Elements of the \tilde{G} matrix will now be derived.

Evaluation of $\int_0^L \phi_{k,x} \phi_{j,x} \gamma_{i,x} dx$.

For $\underline{i = 1}$,

$$\gamma_{i,x} = \gamma_{1,x} = \frac{d}{dx} \left(1 - \frac{x}{L} \right) = -\frac{1}{L}$$

$$\int_0^L \phi_{k,x} \phi_{j,x} \gamma_{1,x} dx = -\frac{1}{L} \int_0^L \phi_{k,x} \phi_{j,x} dx$$

when $k = 1, j = 1$,

$$\begin{aligned}
\phi_{1,x} &= \frac{d}{dx} \left(1 - 3 \frac{x^2}{L^2} + 2 \frac{x^3}{L^3} \right) = -\frac{6x}{L^2} + \frac{6x^2}{L^3} \\
-\frac{1}{L} \int_0^L \left(-\frac{6x}{L^2} + \frac{6x^2}{L^3} \right)^2 dx &= -\frac{1}{L} \int_0^L \left(\frac{36x^2}{L^4} - \frac{72x^3}{L^5} + \frac{36x^4}{L^6} \right) dx \\
&= -\frac{1}{L} \left(\frac{12x^3}{L^4} - \frac{18x^4}{L^5} + \frac{36}{5} \frac{x^5}{L^6} \right) \Big|_0^L = -\frac{1}{L} \left[\frac{12L^3}{L^4} - \frac{18L^4}{L^5} + \frac{36}{5} \frac{L^5}{L^6} \right] \\
&= -\frac{12}{L^2} + \frac{18}{L^2} - \frac{36}{5L^2} = \left(-\frac{60}{5} + \frac{90}{5} - \frac{36}{5} \right) \frac{1}{L^2} = -\frac{6}{5L^2} .
\end{aligned}$$

For $\underline{i = 2}$,

$$\begin{aligned}
\gamma_{2,x} &= \frac{d}{dx} \left(\frac{x}{L} \right) = \frac{1}{L} \\
\int_0^L \phi_{k,x} \phi_{j,x} \gamma_{2,x} dx &= - \int_0^L \phi_{k,x} \phi_{j,x} \gamma_{1,x} dx .
\end{aligned}$$

Other values of j and k are evaluated similarly to yield

$$\int_0^L \phi_{k,x} \phi_{j,x} \gamma_{1,x} dx = \begin{bmatrix} w_1 & w_2 & \theta_1 & \theta_2 \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} & -\frac{4}{30} & \frac{1}{30} \\ \frac{1}{10L} & -\frac{1}{10L} & \frac{1}{30} & -\frac{4}{30} \end{bmatrix} = \left[\phi_k \phi_j \gamma_1 \right] \quad (53)$$

$$\int_0^L \phi_{k,x} \phi_{j,x} \gamma_{2,x} dx = \begin{bmatrix} w_1 & w_2 & \theta_1 & \theta_2 \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} & \frac{4}{30} & -\frac{1}{30} \\ -\frac{1}{10L} & \frac{1}{10L} & -\frac{1}{30} & \frac{4}{30} \end{bmatrix} = [\phi_k \phi_j \gamma_2] \quad (54)$$

Evaluation of $\int_0^L \phi_{i,x} \phi_{j,x} \gamma_{k,x} dx$.

For $\underline{i = 1}$,

$$\phi_{i,x} = \phi_{1,x} = -\frac{6x}{L^2} + \frac{6x^2}{L^3}$$

$$j = 1, k = 1$$

$$\int_0^L \phi_{1,x} \phi_{1,x} \gamma_{1,x} dx = -\frac{6}{5L^2}$$

when $j = 1, k = 2$,

$$\int_0^L \left(-\frac{6x}{L^2} + \frac{6x^2}{L^3} \right) \left(-\frac{6x}{L^2} + \frac{6x^2}{L^3} \right) \left(-\frac{1}{L} \right) dx = \frac{6}{5L^2}$$

$$\int_0^L \phi_{1,x} \phi_{k,x} \gamma_{j,x} dx = \begin{bmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} \\ \frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} \end{bmatrix} = [\phi_1 \phi_k \gamma_j] \quad (55)$$

$$\int_0^L \phi_{2,x} \phi_{k,x} \gamma_{j,x} dx = \begin{bmatrix} \frac{6}{5L^2} & -\frac{6}{5L^2} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} \\ -\frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \end{bmatrix} = \begin{bmatrix} \phi_2 \phi_k \gamma_j \end{bmatrix} \quad (56)$$

$$\int_0^L \phi_{3,x} \phi_{k,x} \gamma_{j,x} dx = \begin{bmatrix} \frac{1}{10L} & -\frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \\ -\frac{4}{30} & \frac{4}{30} \\ \frac{1}{30} & -\frac{1}{30} \end{bmatrix} = \begin{bmatrix} \phi_3 \phi_k \gamma_j \end{bmatrix} . \quad (57)$$

$$\int_0^L \phi_{4,x} \phi_{k,x} \gamma_{j,x} dx = \begin{bmatrix} \frac{1}{10L} & -\frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \\ \frac{1}{30} & -\frac{1}{30} \\ -\frac{4}{30} & \frac{4}{30} \end{bmatrix} = \begin{bmatrix} \phi_4 \phi_k \gamma_j \end{bmatrix} . \quad (58)$$

Evaluation of $\int_0^L \gamma_{k,x} \gamma_{j,x} \gamma_{i,x} dx$.

$$\gamma_{1,x} = -\frac{1}{L} \quad , \quad \gamma_{2,x} = \frac{1}{L}$$

Then,

$$\int_0^L \gamma_{k,x} \gamma_{j,x} \gamma_{1,x} dx = \begin{bmatrix} -\frac{1}{L^2} & \frac{1}{L^2} \\ \frac{1}{L^2} & -\frac{1}{L^2} \end{bmatrix} = [\gamma_k \gamma_j \gamma_1] \quad (59)$$

$$\int_0^L \gamma_{k,x} \gamma_{j,x} \gamma_{2,x} dx = \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} = [\gamma_k \gamma_j \gamma_2] \quad (60)$$

Evaluation of $\int_0^L \phi_{i,x} \phi_{j,x} \gamma_{k,x} dx$.

$$\int_0^L \phi_{i,x} \phi_{j,x} \gamma_{k,x} dx = [\phi_i \phi_j \gamma_k] = [\phi_i \phi_k \gamma_j]^T \quad (61)$$

The above matrices are now multiplied by the q_k^0 state displacements as indicated in equation (46) ,

$$w_k^0 \int_0^L \phi_{k,x} \phi_{j,x} \gamma_{1,x} dx = \begin{bmatrix} w_1^0 & w_2^0 & \theta_1^0 & \theta_2^0 \end{bmatrix} \begin{bmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} & -\frac{4}{30} & \frac{1}{30} \\ \frac{1}{10L} & -\frac{1}{10L} & \frac{1}{30} & -\frac{4}{30} \end{bmatrix} \quad (62)$$

$$= [w^0] \begin{bmatrix} \phi_k \phi_j \gamma_1 \end{bmatrix}$$

$$w_k^0 \int_0^L \phi_{k,x} \phi_{j,x} \gamma_{2,x} dx = \begin{bmatrix} w_1^0 & w_2^0 & \theta_1^0 & \theta_2^0 \end{bmatrix} \begin{bmatrix} \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} & \frac{4}{30} & -\frac{1}{30} \\ -\frac{1}{10L} & \frac{1}{10L} & -\frac{1}{30} & \frac{4}{30} \end{bmatrix}$$

$$= [w^0] \begin{bmatrix} \phi_k & \phi_j & \gamma_2 \end{bmatrix} \quad (63)$$

Similarly,

$$w_k^0 \int_0^L \phi_{1,x} \phi_{k,x} \gamma_{j,x} dx = \begin{bmatrix} w_1^0 & w_2^0 & \theta_1^0 & \theta_2^0 \end{bmatrix} \begin{bmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} \\ \frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} \end{bmatrix} = [w^0] \begin{bmatrix} \phi_1 & \phi_k & \gamma_j \end{bmatrix} \quad (64)$$

$$w_k^0 \int_0^L \phi_{2,x} \phi_{k,x} \gamma_{j,x} dx = [w^0] \begin{bmatrix} \phi_2 & \phi_k & \gamma_j \end{bmatrix} \quad (65)$$

$$w_k^0 \int_0^L \phi_{3,x} \phi_{k,x} \gamma_{j,x} dx = [w^0] \begin{bmatrix} \phi_3 & \phi_k & \gamma_j \end{bmatrix} \quad (66)$$

$$w_k^0 \int_0^L \phi_{4,x} \phi_{k,x} \gamma_{j,x} dx = [w^0] \begin{bmatrix} \phi_4 & \phi_k & \gamma_j \end{bmatrix} \quad (67)$$

$$\begin{aligned}
u_k^0 \int_0^L \phi_{1,x} \phi_{j,x} \gamma_{k,x} dx &= [u_1^0 \ u_2^0] \begin{bmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \end{bmatrix} \\
&= [u^0] [\phi_1 \ \phi_j \ \gamma_k]
\end{aligned} \tag{68}$$

$$u_k^0 \int_0^L \phi_{2,x} \phi_{j,x} \gamma_{k,x} dx = [u^0] [\phi_2 \ \phi_j \ \gamma_k] \tag{69}$$

$$u_k^0 \int_0^L \phi_{3,x} \phi_{j,x} \gamma_{k,x} dx = [u^0] [\phi_3 \ \phi_j \ \gamma_k] \tag{70}$$

$$u_k^0 \int_0^L \phi_{4,x} \phi_{j,x} \gamma_{k,x} dx = [u^0] [\phi_4 \ \phi_j \ \gamma_k] \tag{71}$$

$$\begin{aligned}
u_k^0 \int_0^L \gamma_{k,x} \gamma_{j,x} \gamma_{1,x} dx &= [u_1^0 \ u_2^0] \begin{bmatrix} -\frac{1}{L^2} & \frac{1}{L^2} \\ \frac{1}{L^2} & -\frac{1}{L^2} \end{bmatrix} = [u^0] [\gamma_k \ \gamma_j \ \gamma_1]
\end{aligned} \tag{72}$$

$$u_k^0 \int_0^L \gamma_{k,x} \gamma_{j,x} \gamma_{2,x} dx = [u^0] [\gamma_k \ \gamma_j \ \gamma_2] \tag{73}$$

The above matrices are combined to form the \tilde{G} matrix (Equation (46)):

$$\begin{array}{c}
\begin{array}{cccccc}
u_1^1 & u_2^1 & w_1^1 & w_2^1 & \theta_1^1 & \theta_2^1
\end{array} \\
[G] = \begin{bmatrix}
3EA[u^0] \begin{bmatrix} \gamma_k & \gamma_j & \gamma_1 \end{bmatrix} & \parallel & EA[w^0] \begin{bmatrix} \phi_k & \phi_j & \gamma_1 \end{bmatrix} \\
3EA[u^0] \begin{bmatrix} \gamma_k & \gamma_j & \gamma_2 \end{bmatrix} & \parallel & EA[w^0] \begin{bmatrix} \phi_k & \phi_j & \gamma_2 \end{bmatrix} \\
EA[w^0] \begin{bmatrix} \phi_1 & \phi_k & \gamma_j \end{bmatrix} & \parallel & EA[u^0] \begin{bmatrix} \phi_1 & \phi_j & \gamma_k \end{bmatrix} \\
EA[w^0] \begin{bmatrix} \phi_2 & \phi_k & \gamma_j \end{bmatrix} & \parallel & EA[u^0] \begin{bmatrix} \phi_2 & \phi_j & \gamma_k \end{bmatrix} \\
EA[w^0] \begin{bmatrix} \phi_3 & \phi_k & \gamma_j \end{bmatrix} & \parallel & EA[u^0] \begin{bmatrix} \phi_3 & \phi_j & \gamma_k \end{bmatrix} \\
EA[w^0] \begin{bmatrix} \phi_3 & \phi_k & \gamma_j \end{bmatrix} & \parallel & EA[u^0] \begin{bmatrix} \phi_4 & \phi_j & \gamma_k \end{bmatrix}
\end{bmatrix}
\end{array} \tag{74}$$

Load Behavior Matrix. The effect of applied load behavior on the element stiffness is obtained by adjusting the \tilde{K} matrix with the \tilde{L} matrix, where

$$[L] = [L_p] + [L_F] . \tag{75}$$

The \tilde{L} matrix will now be derived from equations (47) through (51) for each of the three load behavior cases.

Case I.

$$\begin{aligned}
\int_0^L p w_j^1 \phi_{j,x} \gamma_1 dx &= \int_L p \left(1 - \frac{x}{L}\right) \left[w_1^1 \left(-\frac{6x}{L^2} + \frac{6x^2}{L^3} \right) + \theta_1^1 \left(-1 + 4\frac{x}{L} - 3\frac{x^2}{L^2} \right) \right. \\
&\quad \left. + w_2^1 \left(\frac{6x}{L^2} - \frac{6x^2}{L^3} \right) + \theta_2^1 \left(\frac{2x}{L} - 3\frac{x^2}{L^2} \right) \right] dx \\
&= -\frac{1}{2} p w_1^1 + \frac{1}{2} p w_2^1 - \frac{1}{12} p L \theta_1^1 + \frac{1}{12} p L \theta_2^1 .
\end{aligned}$$

$$\int_0^L p w_j^1 \phi_{j,x} \gamma_2 dx = \frac{1}{2} p w_1 + \frac{1}{2} p w_2 + \frac{1}{12} p L \theta_1 - \frac{1}{12} p L \theta_2 \quad .$$

Then,

$$\begin{bmatrix} L \\ p \end{bmatrix} = p \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{L}{12} & \frac{L}{12} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{L}{12} & -\frac{L}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad . \quad (76)$$

\tilde{L}_F may be written directly from equation (48) in view of equations (38) and (39);

$$[L_F] = p \begin{bmatrix} 0 & 0 & 0 & 0 & -\vec{F}_{z1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\vec{F}_{z2} \\ 0 & 0 & 0 & 0 & \vec{F}_{x1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \vec{F}_{x2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad (77)$$

where \vec{F} 's are proportionality constants between the indicated forces and p . The magnitudes of the applied forces are assumed to remain in a constant ratio to each other and to the lateral load, p , during loading of the element. This

assumption is made here and in the Stability Criterion section for illustrative purposes. The assumption is easily altered to study buckling under pressure with specified nodal forces or buckling under nodal forces with specified pressure.

Case II.

$$[L_p] = [L_F] = [0] \quad (78)$$

Case III.

$$\begin{aligned} \int_L \frac{p}{R} u_j^1 \gamma_j \gamma_1 dx &= \int_L \frac{p}{R} \left[u_1^1 \left(1 - \frac{x}{L} \right) + u_2^1 \left(\frac{x}{L} \right) \right] \left[1 - \frac{x}{L} \right] dx \\ &= \frac{p}{R} \int_L \left[\left(1 - \frac{2x}{L} + \frac{x^2}{L^2} \right) u_1^1 + \left(\frac{x}{L} - \frac{x^2}{L^2} \right) u_2^1 \right] dx \\ &= \frac{p}{R} \left[\left(x - \frac{x^2}{L} + \frac{x^3}{3L^2} \right) u_1^1 + \left(\frac{x^2}{2L} - \frac{x^3}{3L^2} \right) u_2^1 \right]_0^L \\ &= \frac{p}{R} \left[\left(L - L + \frac{L}{3} \right) u_1^1 + \left(\frac{L}{2} - \frac{L}{3} \right) u_2^1 \right] \\ &= \frac{p}{R} \left[\frac{L}{3} u_1^1 + \frac{L}{6} u_2^1 \right] \\ \int_L \frac{p}{R} u_j^1 \gamma_j \gamma_2 dx &= \frac{pL}{R} \left[\frac{1}{6} u_1^1 + \frac{1}{3} u_2^1 \right] \end{aligned}$$

$$[L_p] = p \begin{bmatrix} -\frac{L}{3R} & -\frac{L}{6R} & 0 & 0 & 0 & 0 \\ -\frac{L}{6R} & -\frac{L}{3R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (79)$$

\tilde{L}_F may be written directly from equation (51);

$$[L_F] = p \begin{bmatrix} \frac{\vec{F}_{z1}}{R} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\vec{F}_{z2}}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\vec{F}_{x1}}{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\vec{F}_{x2}}{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \quad (80)$$

Coordinate Transformation

The developments thus far have been in a coordinate system which was oriented so that the x-axis coincides with the element centerline. To combine several elements for solution of a particular problem, it is necessary to obtain the stiffness matrices of the elements in a common or global coordinate system. This system will be denoted by \bar{x} , \bar{z} as shown in Figure 5.

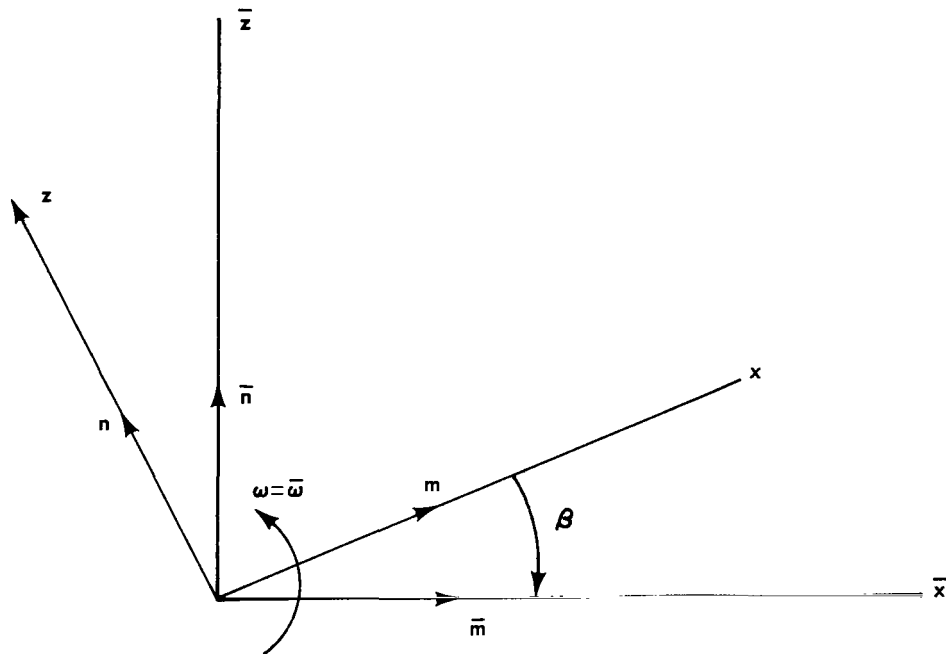


FIGURE 5. COORDINATE TRANSFORMATION

The figure shows that vectors in the two coordinate systems transform according to the relationships

$$m = \bar{m} \cos \beta + \bar{n} \sin \beta$$

$$n = \bar{n} \cos \beta - \bar{m} \sin \beta$$

$$\omega = \bar{\omega} .$$

Writing in matrix notation, we see that

$$\begin{Bmatrix} m \\ n \\ \omega \end{Bmatrix} = \begin{bmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{m} \\ \bar{n} \\ \bar{\omega} \end{Bmatrix}$$

or

$$\{m\} = [T] \{\bar{m}\} . \quad (81)$$

Thus the displacements and forces of the element transform according to

$$\begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ \hline u_2 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{\theta}_1 \\ \hline \bar{u}_2 \\ \bar{w}_2 \\ \bar{\theta}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x1} \\ F_{z1} \\ M_1 \\ \hline F_{x2} \\ F_{z2} \\ M_2 \end{Bmatrix} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{F}_{x1} \\ \bar{F}_{z1} \\ \bar{M}_1 \\ \hline \bar{F}_{x2} \\ \bar{F}_{z2} \\ \bar{M}_2 \end{Bmatrix}$$

or

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} \quad (82)$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix} . \quad (83)$$

If the nodal forces and displacements are related by the stiffness \tilde{K}^1 (equation (44)), then

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K^1] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (84)$$

where \tilde{K}^1 has been rearranged to conform to the \tilde{Q} and \tilde{q} matrices. Upon substituting equations (82) and (83) into equation (84),

$$\begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix} = [K^1] \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix}^{-1} [K^1] \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix} \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} .$$

But in this case,

$$\begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix}^{-1} = \begin{bmatrix} T & | & 0 \\ \hline 0 & | & T \end{bmatrix}^T \quad (85)$$

$$\begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix} = \begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix} [K^1] \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} \quad (86)$$

$$\begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix} = [\bar{K}] \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} \quad (87)$$

where \bar{K} is the stiffness matrix of the element in the global coordinate system.

Master Stiffness Matrix

The stiffness matrix for complete structure is obtained by combining the element stiffness matrices in the global coordinate system. This section describes the method of accomplishing this combination.

Consider part of an overall structure schematically represented by Figure 6. The \bar{Q}_i 's in Figure 6 are intended to represent the total load vector

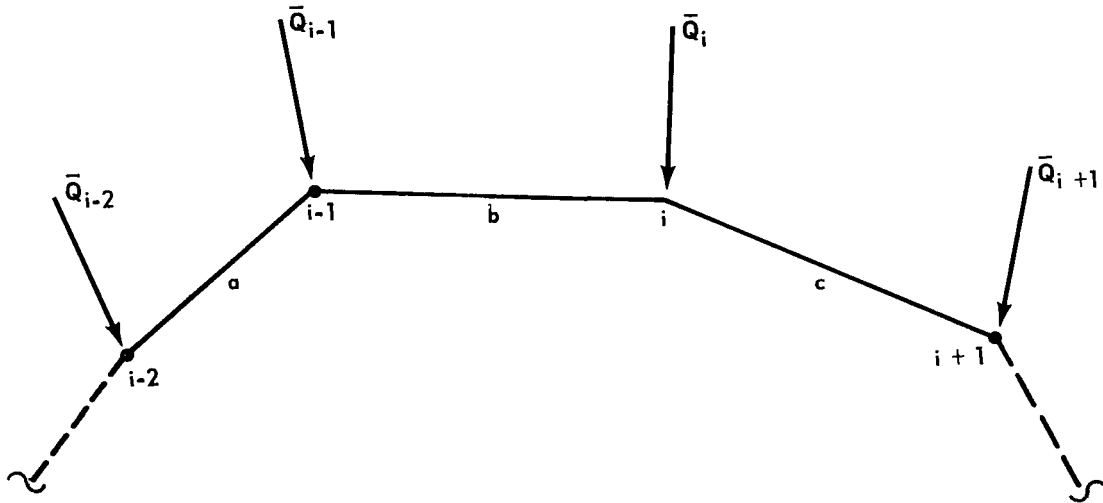


FIGURE 6. GENERAL STRUCTURE

at the point of application,

$$\bar{Q}_i = \begin{Bmatrix} \bar{F}_{xi} \\ \bar{F}_{zi} \\ \bar{M}_i \end{Bmatrix} . \quad (88)$$

Load-displacement relationships for each of the three elements in the global coordinate system are expressed from equation (87) as follows:

$$\begin{Bmatrix} \bar{Q}_{ai-2} \\ \bar{Q}_{ai-1} \end{Bmatrix} = \begin{bmatrix} \bar{K}_{ai-2, i-2} & \bar{K}_{ai-2, i-1} \\ \bar{K}_{ai-1, i-2} & \bar{K}_{ai-1, i-1} \end{bmatrix} \begin{Bmatrix} \bar{q}_{i-2} \\ \bar{q}_{i-1} \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{Q}_{bi-1} \\ \bar{Q}_{bi} \end{Bmatrix} = \begin{bmatrix} \bar{K}_{bi-1, i-1} & \bar{K}_{bi-1, i} \\ \bar{K}_{bi, i-1} & \bar{K}_{bi, i} \end{bmatrix} \begin{Bmatrix} \bar{q}_{i-1} \\ \bar{q}_i \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{Q}_{ci} \\ \bar{Q}_{ci+1} \end{Bmatrix} = \begin{bmatrix} \bar{K}_{ci, i} & \bar{K}_{ci, i+1} \\ \bar{K}_{ci+1, i} & \bar{K}_{ci+1, i+1} \end{bmatrix} \begin{Bmatrix} \bar{q}_i \\ \bar{q}_{i+1} \end{Bmatrix} .$$

By combining the above relationships with the fact that \bar{q}_i is the same for both elements b and c, and noting that

$$\begin{Bmatrix} \bar{Q}_i \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_{bi} \end{Bmatrix} + \begin{Bmatrix} \bar{Q}_{ci} \end{Bmatrix} ,$$

and that similar relationships hold for the other nodes, the master stiffness matrix is obtained:

$$\begin{pmatrix} \bar{Q}_{i-1} \\ \bar{Q}_i \\ \bar{Q}_{i+1} \end{pmatrix} = \begin{bmatrix} \text{---} & \text{---} & 0 & 0 & 0 \\ \text{---} & \bar{K}_{ai-i,i-1} + \bar{K}_{bi-1,i-1} & \bar{K}_{bi-1,i} & 0 & 0 \\ 0 & \bar{K}_{bi,i-1} & \bar{K}_{bii} + \bar{K}_{cii} & \bar{K}_{ci,i+1} & 0 \\ 0 & 0 & \bar{K}_{ci+1,i} & \bar{K}_{ci+1,i+1} + \bar{K}_{di+1,i+1} & \text{---} \\ 0 & 0 & 0 & \text{---} & \text{---} \end{bmatrix} \begin{pmatrix} \bar{q}_{i-1} \\ \bar{q}_i \\ \bar{q}_{i+1} \end{pmatrix} \quad (89)$$

Note that this is a relationship between the externally applied loads and the nodal displacements of the assembled structure in the global coordinate system.

Stability Criterion

Elements with stiffness matrices of the type given by equation (44) may be assembled into a master stiffness matrix to represent a structure subjected to the critical (buckling) magnitude of applied loads. Thus,

$$\left[[\bar{K}] + p [\bar{K}_0] \right] \{ \bar{q}_1 \} = \{ 0 \} \quad (90)$$

where,

$$p [\bar{K}_0] = [\bar{G}] + [\bar{L}_p] + [\bar{L}_F] \quad , \quad (91)$$

and the null matrix of applied external loads indicates that the structural stiffness has vanished under the critical load magnitude (a physical interpretation of instability). Boundary conditions may be applied to reduce the size of the set of equations (90) and the new reduced set again denoted by the equations (90).

The \bar{K} , \bar{G} , and \bar{L}_p matrices above are obtained by assembly of element matrices as described in the preceding sections. The \bar{L}_F matrix is more conveniently obtained by direct application of equations (48) and (51) to each node of the assembled structure in the global coordinate system.

A nontrivial solution of equations (90) will exist only when the determinant of the matrix $\bar{K} + p \bar{K}_0$ vanishes;

$$\left| \bar{K} + p \bar{K}_0 \right| = 0 \quad . \quad (92)$$

This is an eigenvalue problem where the magnitudes of applied load, p , at which instability will occur are the eigenvalues.

Primary Equilibrium State

It should be noted at this point that the foregoing solution is contingent upon a knowledge of the primary equilibrium state. That is, the G matrix cannot be formulated until the q^0 displacements are known in terms of the applied loads. Given a set of applied loads, the nonlinear equilibrium equations (30) through (35) may be solved for the q^0 displacements. An iterative procedure for incorporating this solution into the stability problem is given in Reference 14.

Another approach which is consistent with many classical stability problem solutions is to linearize equations (30) through (35) for the purpose of obtaining the primary state solution. If this is done these equations become for all three cases

$$\int_L EA u_j^0 \gamma_{j,x} \gamma_{i,x} dx - Q_{xi} = 0 \quad (93)$$

$$\int_L \left[EI w_j^0 \phi_{i,xx} \phi_{j,xx} - p \phi_i \right] dx - Q_{zi} = 0 \quad (94)$$

All terms in the above equations have previously been evaluated and expressed in matrix form except the $p \phi_i$ term which is as follows:

$$\begin{aligned} \int_0^L p \phi_1 dx &= \int_0^L p \left(1 - 3 \frac{x^2}{L^2} + 2 \frac{x^3}{L^3} \right) dx \\ &= p \left(x - \frac{x^3}{L^2} + \frac{1}{2} \frac{x^4}{L^3} \right)_0^L = p \left(L - L + \frac{1}{2} L \right) = \frac{1}{2} pL, \\ \int_0^L p \phi_2 dx &= \frac{1}{2} pL, \\ \int_0^L p \phi_3 dx &= \int_0^L p \left(-x + \frac{2x^2}{L} - \frac{x^3}{L^2} \right) dx = p \left(-\frac{x^2}{2} + \frac{2}{3} \frac{x^3}{L} - \frac{1}{4} \frac{x^4}{L^2} \right)_0^L \\ &= p \left(-\frac{L^2}{2} + \frac{2}{3} L^2 - \frac{1}{4} L^2 \right) = -\frac{1}{12} pL^2 \end{aligned}$$

$$\int_0^L p \phi_4 dx = \frac{1}{12} p L^2 \quad L$$

or

$$\{p\phi\} = p \begin{pmatrix} 0 \\ 0 \\ \frac{L}{2} \\ \frac{L}{2} \\ -\frac{L^2}{12} \\ \frac{L^2}{12} \end{pmatrix} \quad (95)$$

Equations (93, 94) may then be expressed in the form

$$[K] \{q^0\} = \{p\phi\} + \{Q\} \quad (96)$$

From this expression for each element, the master stiffness matrix for the entire structure is assembled as before. Then for the complete structure,

$$\{\bar{q}^0\} = [K]^{-1} \{\bar{p}\phi + \bar{Q}\} \quad (97)$$

A further simplification for obtaining q^0 which has also been used extensively in classical problems is the use of a membrane solution for q^0 . The structure is assumed to take no loads in bending before buckling. For many practical problems such a solution can readily be obtained by inspection, elementary equilibrium considerations, or from the literature. For simplicity this approach is used in the example problem of this report.

APPLICATION OF THE THEORY

This section is intended to give a step by step approach for applying the theory to the solution of practical problems. For problems with several elements

either a general computer program would need to be developed or a relatively simple computer program for each application could be developed as was done for the example to follow later.

One begins by dividing the structure to be analyzed into discrete elements, the number depending upon the desired accuracy. From the known properties and loading on each element the \tilde{K} matrix can be calculated from equation (52) and the \tilde{L}_p and \tilde{L}_F matrices determined from the appropriate equations (76), (77), (78), (79), or (80). Note that the critical magnitude, p , of the applied loading remains as the unknown to be determined.

The q^0 primary state is now determined from a known membrane solution or from equation (97). The \tilde{K} matrix in equation (97) is assembled by using equations (81) and (86) to transform elements to the global coordinate system and equation (89) for combining the elements. Boundary conditions are applied and equation (97) is solved for q^0 . Once the q^0 state is determined the G matrix can be found from equation (74), using the definitions (53) through (61).

The \tilde{K}^1 matrix defined by equation (44) is now known for each element. These are assembled into a master stiffness matrix by equations (86) and (89), and boundary conditions are applied to the resulting set of equations to obtain equations (90). Eigenvalues of the characteristic determinant, equations (92), which is obtained from equation (90), are the magnitudes of the buckling load. The mode shape for each buckling load can be determined by substituting each eigenvalue in turn into equations (90) and solving for the relative amplitudes of displacements, \bar{q}_i .

EXAMPLE PROBLEM

Circular Arch With Uniform Pressure

The above application procedure was applied to the uniform circular arch shown in Figure 7. The known membrane solution for a circular arch under uniform pressure is

$$\begin{aligned}
[q^0] &= [u^0 \ w^0] = \begin{bmatrix} u_1^0 & u_2^0 & w_1^0 & w_2^0 & \theta_1^0 & \theta_2^0 \end{bmatrix} \\
&= \frac{pR}{AE} \begin{bmatrix} \frac{L}{2} & -\frac{L}{2} & -R & -R & 0 & 0 \end{bmatrix}
\end{aligned} \tag{98}$$

for all elements.

The \underline{G} matrix is formulated as indicated by equation (74) :

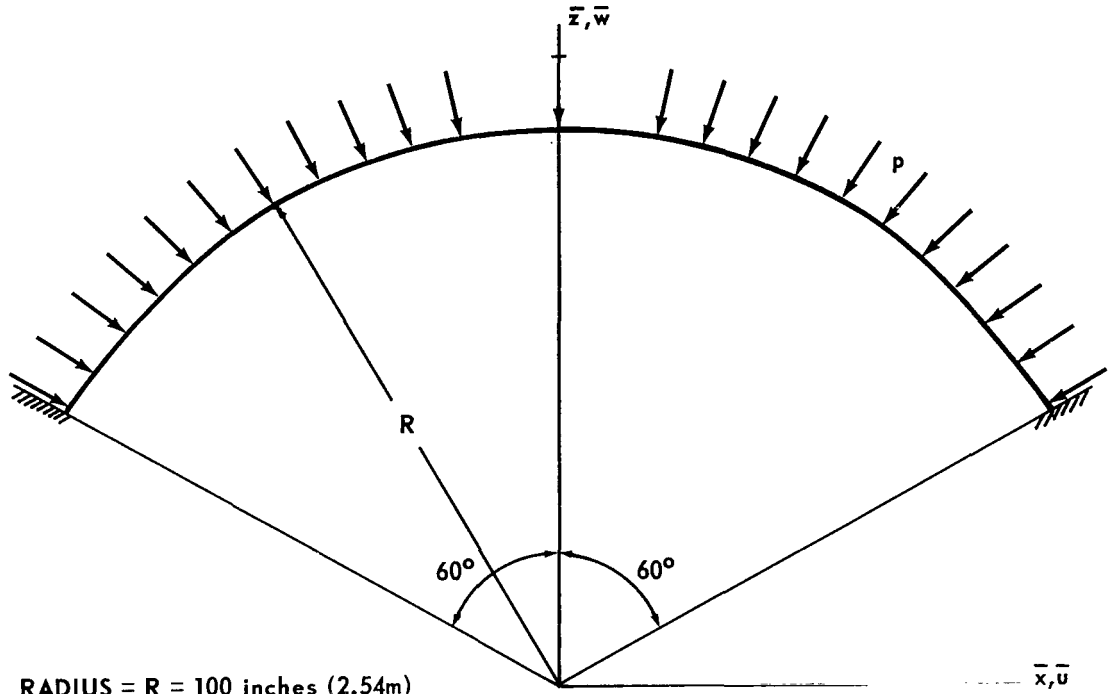
$$3EA [u^0] \begin{bmatrix} \gamma_k \gamma_j \gamma_1 \end{bmatrix} = 3EA \left(\frac{pR}{AE} \right) \begin{bmatrix} \frac{L}{2} & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{L^2} & \frac{1}{L^2} \\ \frac{1}{L^2} & -\frac{1}{L^2} \end{bmatrix}$$

$$= \frac{3pR}{L} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$3EA [u^0] \begin{bmatrix} \gamma_k \gamma_j \gamma_2 \end{bmatrix} = \frac{3pR}{L} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$EA [u^0] \begin{bmatrix} \phi_1 \phi_j \gamma_k \end{bmatrix} = EA \left(\frac{pR}{AE} \right) \begin{bmatrix} \frac{L}{2} & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \end{bmatrix}$$

$$= pR \begin{bmatrix} -\frac{6}{5L} & \frac{6}{5L} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$$



RADIUS = $R = 100$ inches (2.54m)
 AREA = $A = 0.628318$ in² (4.05 cm²)
 MOMENT OF INERTIA = $I = 0.314159$ in⁴ (13.1 cm⁴)
 YOUNG'S MODULUS = $E = 10^7$ psi (6.9×10^{10} N/m²)

FIGURE 7. UNIFORM CIRCULAR ARCH

Other elements of the \tilde{G} matrix are calculated similarly to yield for all elements

$$[G] = pR \begin{bmatrix}
 \frac{3}{L} & \frac{3}{L} & 0 & 0 & 0 & 0 \\
 \frac{3}{L} & -\frac{3}{L} & 0 & 0 & 0 & 0 \\
 0 & 0 & -\frac{6}{5L} & \frac{6}{5L} & \frac{1}{10} & \frac{1}{10} \\
 0 & 0 & \frac{6}{5L} & -\frac{6}{5L} & -\frac{1}{10} & -\frac{1}{10} \\
 0 & 0 & \frac{1}{10} & -\frac{1}{10} & -\frac{4L}{30} & \frac{L}{30} \\
 0 & 0 & \frac{1}{10} & -\frac{1}{10} & \frac{L}{30} & -\frac{4L}{30}
 \end{bmatrix} \begin{matrix} F_{x1} \\ F_{x2} \\ F_{z1} \\ F_{z2} \\ M_1 \\ M_2 \end{matrix} \quad (99)$$

where the four terms in the upper left corner can be traced directly to the $u_{,x}^2$ term in equation (11).

Now for all elements the \tilde{K} matrix is given by equation (52), the \tilde{L}_p matrices for the three load cases are given by equations (76), (78), and (79), respectively, and all \tilde{L}_F matrices are null.

The \tilde{K} , \tilde{G} , and \tilde{L}_F matrices are now all rearranged to the order,

$$[K^1] \equiv \begin{array}{c} \begin{array}{ccccc} u_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_2 \end{array} \\ \left[\begin{array}{ccc|ccc} & & & & & \\ & K_{11}^1 & & & K_{12}^1 & \\ & & & & & \\ \hline & & & & & \\ & K_{21}^1 & & & K_{22}^1 & \\ & & & & & \end{array} \right] \begin{array}{l} F_{x1} \\ F_{z1} \\ M_1 \\ F_{x2} \\ F_{z2} \\ M_2 \end{array} \end{array} \quad (100)$$

All element matrices are now rotated to the global coordinate system, \bar{x} , \bar{y} by the operation,

$$\begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix} [K^1] \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = [\bar{K}] \quad (101)$$

as indicated by equation (86). The T matrix is of course calculated separately for each element as indicated by equation (81) where β is the angle measured clockwise from the element x axis to the \bar{x} axis.

The element matrices are now combined to form the master stiffness matrix as indicated by equation (89). The size of this matrix is reduced by applying the boundary conditions,

$$u_1 = w_1 = \theta_1 = u_2 = w_2 = \theta_2 = 0, \quad (102)$$

where u_1 , w_1 , θ_1 are displacements of the right end of the last element. From

this reduced matrix the characteristic determinant (92) is formed and eigenvalues are calculated. The above operations were performed on a computer for idealizations of 2, 3, 6, 9, and 12 elements. A hand calculation was also made for the two-element arch. Results of the analysis are shown in Table I. As can be seen from the table, comparison with known results is excellent and there is rapid convergence to the exact solutions. The exact solutions were obtained from the analysis by Wempner and Kesti; [24]. The 12-element solution for Case II has also been previously obtained by the finite element method in Reference 14.

TABLE I. COMPARISON OF ARCH BUCKLING LOADS

Number of Elements	Buckling Pressure (lb/in)		
	Case I	Case II	Case III
2	94.1	94.1	94.1
3	57.1	64.7	67.4
6	57.4	62.6	64.4
9	56.6	61.9	63.6
12	56.4	61.6	63.3
Exact solution	56.87	60.95	63.46

CONCLUSIONS

The results of the example problem above show that the theory developed here may be quite useful in solving practical engineering stability problems.

The implicit assumption of uniform shear on the element mentioned in Development of Element Stiffness Matrix section does not appear to adversely affect the results. However, if desired, this assumption could possibly be eliminated by the technique introduced by Pian [19]. Also, the $u_{,x}^2$ term of equation (11) apparently has a negligible effect on the results for the particular example solved, since the exact solutions used for comparison do not contain effects of this term.

A very important item to note is that the present theory applies to unconservative systems (represented by Case I) as well as conservative systems since the principle of virtual work was used. It is known, however, that certain unconservative systems are stable in the static sense (considered here), but unstable in the dynamic sense. (See Reference 25, page 152 through 156.) Thus the present analysis could not be expected to adequately treat this type of problem, and great care should be used in applications to unconservative systems.

Fruitful extensions of the present theory would likely be found in the area of curved and three-dimensional elements and in a general automation of the theory application.

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Marshall Space Flight Center, Alabama 35812, May 1, 1968
981-10-10-0000-50-00-008

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